

STICKY MATROIDS AND KANTOR'S CONJECTURE

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Dedicated to Achim Bachem on the occasion of his 70th birthday.

ABSTRACT. We prove the equivalence of Kantor's Conjecture and the Sticky Matroid Conjecture due to Poljak und Turzík.

1. INTRODUCTION

The purpose of this paper is to prove the equivalence of two classical conjectures from combinatorial geometry. Kantor's Conjecture [5] addresses the problem whether a combinatorial geometry can be embedded into a modular geometry, i.e. a direct product of projective spaces. He conjectured that this is impossible only if there exists a non-modular pair of hyperplanes.

The other conjecture, the Sticky Matroid Conjecture (SMC) due to Poljak and Turzík [8] concerns the question whether it is possible to glue two matroids together along a common part. They conjecture that a "common part" for which this is always possible, a sticky matroid, must be modular. It is well-known (see eg. [7]) that modular matroids are sticky and easy to see [8] that modularity is necessary for ranks up to three. Bachem and Kern [1] proved that a rank-4 matroid which has two hyperplanes intersecting in a point is not sticky. They also stated that a matroid is not sticky if for each of its non-modular pairs there exists an extension decreasing its modular defect. The proof of this statement had a flaw which was fixed by Bonin [2]. Using a result of Wille [9] and Kantor [5] this implies that the sticky matroid conjecture is true if and only if it holds in the rank-4 case. Bonin [2] also showed that a matroid of rank $r \geq 3$ with two disjoint hyperplanes is not sticky and that non-stickiness is also implied by the existence of a hyperplane and a line which do not intersect but can be made modular in an extension.

We generalize Bonin's result and show that a matroid is not sticky, if it has a non-modular pair which admits an extension decreasing its modular defect. Moreover by showing the existence of the proper amalgam we prove that in the rank-4 case this condition is also necessary for a matroid not to be sticky. As a consequence from every counterexample to Kantor's conjecture arises a matroid, which can be extended in finite steps to a counterexample of the (SMC) implying the equivalence of the two conjectures. A further consequence of our results is the equivalence of both conjectures to the following:

Conjecture 1. *In every finite non-modular matroid there exists a non-modular pair and a point-extension decreasing its modular defect.*

Finally, we present an example proving that the (SMC), like Kantor's Conjecture fails in the infinite case.

We assume familiarity with matroid theory, the standard reference is [7].

2. OUR RESULTS

Let M be a matroid with groundset E and rank function r . We define the *modular defect* $\delta(X, Y)$ of a pair of subsets $X, Y \subseteq E$ as

$$\delta(X, Y) = r(X) + r(Y) - r(X \cup Y) - r(X \cap Y).$$

By submodularity of the rank function the modular defect is always non-negative. If it equals zero, we call (X, Y) a *modular pair*. A matroid is called *modular*, if all pairs of flats form a modular pair.

An *extension* of a matroid M on a set E is a matroid N on a set $F \supseteq E$ such that $M = N|E$. If N_1, N_2 are extensions of a common matroid M with groundsets F_1, F_2 resp. E such that $F_1 \cap F_2 = E$, then a matroid $A(N_1, N_2)$ with groundset $F_1 \cup F_2$ is called an *amalgam* of N_1 and N_2 if $A(N_1, N_2)|F_i = N_i$ for $i = 1, 2$.

Theorem 1 (Ingleton see [7] 11.4.10 (ii)). *If M is a modular matroid then for any pair (N_1, N_2) of extensions of M an amalgam exists.*

We found only a proof of this result for finite matroids (see eg. [7]). We will show that it also holds for infinite matroids of finite rank.

Conjecture 2 (Sticky Matroid Conjecture (SMC) [8]). *If M is a matroid such that for all pairs (N_1, N_2) of extensions of M an amalgam exists, then M is modular.*

The following preliminary results concerning the (SMC) are known:

Theorem 2 ([8, 1, 2]). (i) *If $r(M) \leq 3$ then the (SMC) holds for M .*
(ii) *If the (SMC) holds for all rank-4 matroids M , then it is true in all ranks.*
(iii) *Let l be a line and H a hyperplane in a matroid M such that $l \cap H = \emptyset$. If M has an extension M' such that $r_{M'}(\text{cl}_{M'}(l) \cap \text{cl}_{M'}(H)) = 1$, then M is not sticky.*

We will generalize the last assertion and prove:

Theorem 3. *Let M be a matroid, X and Y two flats such that $\delta(X, Y) > 0$. If M has an extension M' such that $\delta_{M'}(\text{cl}_{M'}(X), \text{cl}_{M'}(Y)) < \delta(X, Y)$ then M is not sticky.*

We postpone the proof of Theorem 3 to Section 3.

We call a matroid *hypermodular*, if each pair of hyperplanes forms a modular pair. With this notion we can rephrase Kantor's Conjecture.

Conjecture 3 (Kantor [5]). *Every finite hypermodular matroid embeds into a modular matroid.*

Like the (SMC) Kantor's Conjecture can be reduced to the rank-4 case (see Corollary 3, Section 5).

Next, we consider the correspondence between one-point extensions of matroids and modular cuts.

Definition 1. *A set \mathcal{M} of flats of a matroid M is called a modular cut of M , if the following holds:*

- (i) *If $F \in \mathcal{M}$ and F' is a flat in M with $F' \supseteq F$, then $F' \in \mathcal{M}$.*
- (ii) *If $F_1, F_2 \in \mathcal{M}$ and (F_1, F_2) is a modular pair, then $F_1 \cap F_2 \in \mathcal{M}$.*

Theorem 4 (Crapo 1965 [3]). *There is a one-to-one-correspondence between the one-point extensions $M \cup p$ of a matroid M and the modular cuts \mathcal{M} of M . \mathcal{M} consists precisely of the set of flats of M containing the new point p in $M \cup p$.*

The set of all flats of a matroid M is a modular cut, the *trivial modular cut*, corresponding to an extension with a loop. The empty set is a modular cut corresponding to an extension with a coloop, the only one-point extension increasing the rank of M . Let F be a flat of M , then $\mathcal{M}_F = \{G \mid G \text{ is a flat of } M \text{ and } G \supseteq F\}$ is a modular cut of M , we call it a *principal modular cut* and say that in the corresponding extension the new point is *freely added* to F . A *modular cut* $\mathcal{M}_\mathcal{A}$ generated by a set of flats \mathcal{A} is the smallest modular cut containing \mathcal{A} .

The following is immediate from Theorem 7.2.3 of [7].

Proposition 1. *If (X, Y) is a non-modular pair of flats of a matroid M , then there exists an extension decreasing its modular defect (we call the pair intersectable) if and only if the modular cut generated by X and Y is not the principal modular cut $\mathcal{M}_{X \cap Y}$.*

We call a matroid *OTE* (only trivially extendable) if all of its modular cuts different from the empty modular cut are principal.

Most of this paper will be devoted to the proof of the following theorem.

Theorem 5. *If M is a finite rank-4 matroid that is OTE, then M is sticky.*

Thus the (SMC) is equivalent to the conjecture that every finite rank-4 matroid which is OTE is already modular. Again, we can reduce the general case to the rank-4 case, resulting in Conjecture 1. That conjecture is no longer true in the infinite case. We will prove the following theorem in Section 5.

Theorem 6. *Every finite matroid can be extended to a (not necessarily finite) matroid of the same rank that is OTE.*

Starting from, say, the Vámos-matroid this yields a counterexample to the (SMC) in the infinite case.

Finally, any finite counterexample to Kantor's Conjecture can be embedded into a finite non-modular matroid that is OTE yielding a counterexample to the (SMC). Since it is known (and immediate from Theorem 3) that Kantor's Conjecture implies the (SMC), this establishes the equivalence of the two conjectures.

Corollary 1. *Kantor's Conjecture holds true if and only if the Sticky Matroid Conjecture holds true.*

3. PROOF OF THEOREM 3

We start with a proposition which states that the so called Escher-matroid ([7] Fig. 1.9) is not a matroid.

Proposition 2. *Let l_1, l_2, l_3 be three lines in a matroid which are pairwise coplanar but not all lying in a plane. If l_1 and l_2 intersect in a point p , then p must also be contained in l_3 .*

Proof. By submodularity of the rank function we have

$$r((l_1 \vee l_3) \wedge (l_2 \vee l_3)) \leq r(l_1 \vee l_3) + r(l_2 \vee l_3) - r(l_1 \vee l_2 \vee l_3) = 3 + 3 - 4 = 2.$$

Now $l_3 \vee p \leq (l_1 \vee l_3) \wedge (l_2 \vee l_3)$ and hence p must lie on l_3 . \square

Probably the easiest way to prove that the (SMC) holds for rank 3 is to proceed as follows. If a rank-3 matroid M is not modular it has a pair of disjoint lines. We consider two extensions N_1 and N_2 . N_1 adds to the two lines a point of intersection

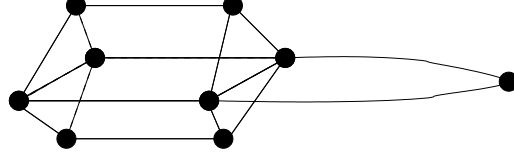


FIGURE 1. This is not a matroid

and N_2 erects a Vámos-cube (V_8 in [7]) using the disjoint lines as base points. By Proposition 2 the amalgam of N_1 and N_2 cannot exist (see Figure 1).

Bonin [2] generalized this idea to the situation of a disjoint line-hyperplane pair in arbitrary dimension. We further generalize this to a non-modular pair of a hyperplane H and a flat F which can be made modular by a proper extension. Our first aim is to show, that such a pair exists in any matroid which is not OTE. Again, the following is immediate:

Proposition 3. *Let M be a matroid, M' an extension of M and (X, Y) a modular pair of flats in M . Then $(\text{cl}_{M'}(X), \text{cl}_{M'}(Y))$ is a modular pair in M' . Moreover*

$$\text{cl}_{M'}(X) \cap \text{cl}_{M'}(Y) = \text{cl}_{M'}(X \cap Y).$$

Proposition 4. *Let M be a matroid, \mathcal{M} a modular cut in M and $M' = M \cup \{p\}$ the corresponding point extension. If M' does not contain a modular pair of flats $X' = X \cup \{p\}, Y' = Y \cup \{p\}$ such that X and Y are a non-modular pair in M , then*

$$\mathcal{M}' := \{\text{cl}_{M'}(F) \mid F \in \mathcal{M}\}$$

is a modular cut.

Lemma 1. *Let M_0 be a matroid which is not OTE and (X, Y) be a non-modular pair of smallest modular defect $\delta := \delta(X, Y)$ such that there is a point extension decreasing their modular defect.*

Then there exists a sequence M_1, \dots, M_δ of matroids such that M_i is a point extension of M_{i-1} for $i = 1, \dots, \delta$ and $\delta_{M_i}(\text{cl}_{M_i}(X), \text{cl}_{M_i}(Y)) = \delta - i$. In particular $(\text{cl}_{M_\delta}(X), \text{cl}_{M_\delta}(Y))$ are a modular pair in M_δ .

Proof. Let \mathcal{M} denote the modular cut generated by X and Y in M_0 . Inductively we conclude, that by the choice of X and Y

$$\mathcal{M}_i := \{\text{cl}_{M_i}(F) \mid F \in \mathcal{M}\}$$

is a modular cut in M_i for $i = 1, \dots, \delta - 1$ implying the assertion. \square

Lemma 2. *Let M be a matroid which is not OTE. Then there exists an intersectable non-modular pair (F, H) of smallest modular defect, where F is a minimal element in the modular cut $\mathcal{M}_{F, H}$ generated by H and F , and H is a hyperplane of M .*

Proof. Since M is not OTE, it is not modular and hence of rank at least three. Every non-modular pair of flats in a rank-3 matroid clearly satisfies the assertion. Hence we may assume $\text{r}(M) \geq 4$. Let (X, Y) be a non-modular intersectable pair of flats in M of smallest modular defect δ_{\min} and chosen such that, furthermore,

X is of minimal and Y of maximal rank. We claim that $F = X$ and $H = Y$ are as required. Let $\mathcal{M}_{X,Y}$ be the modular cut generated by these two flats.

Assume, contrary to the first assertion, that there exists $F \in \mathcal{M}_{X,Y}$ with $X \cap Y \subsetneq F \subsetneq X$. Since $\mathcal{M}_{X,Y}$ contains F and Y but not $F \cap Y = X \cap Y$, the pair (F, Y) is non-modular and intersectable in M (according to Proposition 4). Due to submodularity of r we have $r(X) + r(F \cup Y) \geq r(X \cup Y) + r(F)$ and hence:

$$\begin{aligned} \delta(F, Y) &= r(F) + r(Y) - r(F \cup Y) - r(F \cap Y) \\ &\leq r(X) + r(Y) - r(X \cup Y) - r(X \cap Y) = \delta(X, Y) = \delta_{\min}, \end{aligned}$$

contradicting the choice of X . Next we show that $\text{cl}(X \cup Y) = E(M)$. Assume to the contrary that there exists $p \in E(M) \setminus \text{cl}(X \cup Y)$ and let $Y_1 = \text{cl}(Y \cup \{p\})$. Then $X \cap Y = X \cap Y_1$ and hence $\delta(X, Y_1) = \delta(X, Y)$. Since $\mathcal{M}_{X,Y_1} \subseteq \mathcal{M}_{X,Y}$, the pair (X, Y_1) remains intersectable, contradicting the choice of Y , and hence verifying $\text{cl}(X \cup Y) = E(M)$.

Finally, assume Y is not a hyperplane. Let $Y' = \text{cl}(Y \cup p)$ with $p \in X \setminus Y$. Then

$$\begin{aligned} \delta(X, Y') &= r(X) + r(Y') - r(X \cup Y') - r(X \cap Y') \\ &= r(X) + r(Y) + 1 - r(X \cup Y) - r(X \cap Y) - 1 = \delta(X, Y). \end{aligned}$$

Since Y is not a hyperplane and $\text{cl}(X \cup Y) = E(M)$, we must have $X \cap Y' \subsetneq X$, and X being minimal in $\mathcal{M}_{X,Y}$ implies $X \cap Y' \notin \mathcal{M}_{X,Y}$. Now $\mathcal{M}_{X,Y'} \subseteq \mathcal{M}_{X,Y}$ yields that $X \cap Y' \notin \mathcal{M}_{X,Y'}$ and thus by Proposition 4 the pair (X, Y') is intersectable with $\delta(X, Y') = \delta(X, Y) = \delta_{\min}$, contradicting the choice of Y . \square

Lemmas 1 and 2 now imply the following:

Theorem 7. *Let M be a matroid which is not OTE. Then there exists a non-modular pair (F, H) , where H is a hyperplane of M and an extension N of M such that $(\text{cl}_N(F), \text{cl}_N(H))$ is a modular pair in N .*

On the other hand we also have:

Theorem 8. *Let M be a matroid and (F, H) a non-modular pair of disjoint flats, where H is a hyperplane of M . Then there exists an extension N of M such that for every extension N' of N , $(\text{cl}_{N'}(F), \text{cl}_{N'}(H))$ is not a modular pair in N' .*

Proof. We follow the idea from [1] and Bonin's proof [2] and erect a Vámos-type matroid above F and H .

Clearly, $r(M) \geq 3$ and $2 \leq r_M(F) \leq r - 1$. We extend M by first adding a set A of $r - 1 - r_m(F)$ elements freely to H . Next, we add, first, a coloop e , and then an element f freely to the resulting matroid, yielding an extension N_0 with groundset $E(M) \cup A \cup \{e, f\}$ and of rank $r + 1$.

Note, that $\text{cl}_{N_0}(H) = H \cup A$. We consider the following sets which, consequently, are hyperplanes of N_0 :

- $T_1 = H \cup A \cup e$,
- $T_2 = F \cup A \cup e$,
- $B_1 = H \cup A \cup f$, and
- $B_2 = F \cup A \cup f$.

Note that (T_i, B_i) form a modular pair for $i = 1, 2$.

The sets $\mathcal{M}_T = \{T_1, T_2, E(N_0)\}$ and $\mathcal{M}_B = \{B_1, B_2, E(N_0)\}$ form modular cuts in N_0 . Since F and H are disjoint and a non-modular pair of hyperplanes in a matroid is always intersectable, we find independent sets P and Q , each of size

$\delta(T_1, T_2) = 2r - (r + 1) - (r - r_M(F)) = r_M(F) - 1 = \delta(B_1, B_2)$ such that (T_1, T_2) and (B_1, B_2) become modular pairs as well in the corresponding extension. This way, we obtain a matroid $N = N_{P,Q}$ of rank $r + 1$ with groundset $E(N_0) \cup P \cup Q$. Note that $A \cup P \cup \{e\}$ and $A \cup Q \cup \{f\}$ are independent flats of rank $r - 1$ in N . We will show now, that this matroid N is as required.

Assume to the contrary that there exists an extension N' of N , such that $(\text{cl}_{N'}(F), \text{cl}_{N'}(H))$ is a modular pair in N' . First we observe that as $A \subseteq \text{cl}_{N'}(H)$

$$r_{N'}(F \cup A) - r_{N'}(\text{cl}_{N'}(F \cup A) \wedge \text{cl}_{N'}(H)) \leq r_{N'}(F) - r(\text{cl}_{N'}(F) \wedge \text{cl}_{N'}(H))$$

and so

$$\begin{aligned} & r_{N'}(F \cup A) + r_{N'}(H) - r_{N'}(F \cup H) - r_{N'}(\text{cl}_{N'}(F \cup A) \wedge \text{cl}_{N'}(H)) \\ & \leq r_{N'}(F) + r_{N'}(H) - r_{N'}(F \cup H) - r(\text{cl}_{N'}(F) \wedge \text{cl}_{N'}(H)) = 0. \end{aligned}$$

Hence $(\text{cl}_{N'}(F \cup A), \text{cl}_{N'}(H))$ is a modular pair as well. We use the abbreviations $l_1 = \text{cl}_{N'}(F \cup A)$, $l_2 = \text{cl}_{N'}(H)$, $l_3 = \text{cl}_{N'}(A \cup P \cup \{e\})$. Generalizing Proposition 2 we compute

$$r(l_1 \vee l_3) \wedge (l_2 \vee l_3) \leq r(l_1 \vee l_3) + r(l_2 \vee l_3) - r(l_1 \vee l_2 \vee l_3) = 2r - (r + 1) = r(l_3).$$

Now $l_3 \vee (l_1 \wedge l_2) \leq (l_1 \vee l_3) \wedge (l_2 \vee l_3)$ and hence $(l_1 \wedge l_2) \leq l_3$. The same way we get $(l_1 \wedge l_2) \leq \text{cl}_{N'}(A \cup Q \cup \{f\}) =: l_4$.

We conclude $r - 2 \geq r(l_3 \wedge l_4) \geq r(l_1 \wedge l_2) = r - 2$. Thus, we have equality throughout. On the other hand $r - 1 = r(l_3) < r(A \cup P \cup Q \cup \{e\})$ together with the fact that f is in general position in N implies that $A \cup P \cup Q \cup \{e, f\}$ spans N' and hence $r(l_3 \vee l_4) = r + 1$. This finally yields

$$r(l_3) + r(l_4) = 2r - 2 < (r - 2) + (r + 1) = r(l_3 \wedge l_4) + r(l_3 \vee l_4)$$

contradicting submodularity. \square

Summarizing the two previous theorems yields the final result of this section:

Theorem 9. *Let M be a matroid which is not OTE. Then M is not sticky.*

Proof. By Theorem 7 M has a non-modular intersectable pair of flats (F, H) , such that H is a hyperplane, and there exists an extension N_1 of M such that $(\text{cl}_{N_1}(F), \text{cl}_{N_1}(H))$ is a modular pair. Possibly contracting $(F \cap H)$, and referring to Lemma 7 of [1], we may assume that F and H are disjoint. Thus, by Theorem 8 there also exists an extension N_2 of M such that in every extension $N \supseteq N_2$ the pair $(\text{cl}_N(F), \text{cl}_N(H))$ is not modular. Hence M is not sticky. \square

4. HYPERMODULARITY AND OTE-MATROIDS

We collect some facts about hypermodular matroids and OTE-matroids which we need for the proof of Theorem 5 and the embedding theorems. Recall that a matroid is hypermodular, if any pair of hyperplanes intersects in a coline. Modular matroids are hypermodular and hypermodular matroids of rank at most 3 must be modular. Thus, contractions of hypermodular matroids of rank n by a flat of rank $n - 3$ are modular matroids of rank 3. Every projective geometry $P(n, q)$ is hypermodular as a matroid and remains hypermodular if we delete up to $q - 3$ of its points. In the following we will focus on the case of hypermodular matroids of rank 4.

Proposition 5. *Let M be a hypermodular rank-4 matroid. If M contains a disjoint pair of a line and a hyperplane, then it also contains two disjoint coplanar lines.*

Proof. Let (l_1, e_1) be the disjoint line-plane pair in M . Take a point p in e_1 . Because of hypermodularity, the plane $l_1 \vee p$ intersects the plane e_1 in a line l_2 in M . The lines l_1 and l_2 are coplanar and disjoint. \square

The next results are matroidal versions of similar results of Klaus Metsch [6]) for linear spaces.

Lemma 3. *Let M be a hypermodular matroid of rank 4 on a groundset E . Let l_1, l_2 be two disjoint coplanar lines. Then E can be partitioned into lines disjoint coplanar to l_1 and l_2 .*

Proof. We set $e = \text{cl}(\{l_1 \cup l_2\})$. Then $l_p := (l_1 \vee p) \wedge (l_2 \vee p)$ is a line for every $p \in E \setminus e$ and coplanar to l_1 and l_2 . By Proposition 2 it must be disjoint from l_1 and l_2 . This together with Proposition 2 implies, that for $p \neq q$ we must have either $l_p \wedge l_q = 0$ or $l_p = l_q = p \vee q$. We denote the set of lines constructed this way by Δ .

Now, fix some l_{p^*} for $p^* \in E \setminus e$ and for all $r \in e \setminus (l_1 \cup l_2)$ denote by l_r the line $l_r = (l_{p^*} \vee r) \wedge (l_1 \vee l_2)$. Then l_r must be disjoint from l_1 and l_{p^*} , for otherwise l_1, l_r and l_{p^*} contradict Proposition 2. The same holds for l_2 . Applying Proposition 2 to l_r, l_s and l_{p^*} shows that if $r, s \in e \setminus (l_1 \cup l_2)$, then $l_r \wedge l_s = 0$ or $l_s = l_r$. And finally the triple l_1, l_r, l_q with $l_q \in \Delta$ yield that l_r must be disjoint from l_q and hence from all lines of Δ . We denote the set of lines constructed here by Σ . Hence Δ, Σ, l_1 and l_2 partition the groundset. \square

Lemma 3 and Proposition 5 imply that a non-trivial modular cut in a hypermodular rank-4 matroid contains a set of pairwise disjoint lines that partition the ground set. Moreover we have:

Theorem 10. (i) *Under the assumptions of Lemma 3 there exists a point extension, where l_1 and l_2 intersect, if and only if the lines in the partition are pairwise coplanar.*
(ii) *If a point extension M' as in (i) exists, then the restriction to M of any line in M' is a line.*
(iii) *Otherwise, M contains two non-coplanar lines l_3, l_4 such that l_i and l_j are coplanar for all $i \in \{1, 2\}$ and $j \in \{3, 4\}$ and no three of them are coplanar, i.e. it has the Vámos-matroid, “containing” l_1 and l_2 as a restriction.*

Proof. (i) Since all the lines in the partition were constructed using modular intersections of hyperplanes that must be contained in any modular cut containing l_1 and l_2 they must all be contained in the modular cut generated by l_1 and l_2 . Hence any two lines from the partition intersect in the new point, implying that they must be pairwise coplanar.

On the other hand, if all the lines in l_1, l_2, Δ and Σ are pairwise coplanar, then these lines form the minimal elements of a modular cut \mathcal{M} . This is seen as follows. Consider the upset of these lines. Any two lines in that set are disjoint and coplanar, hence they do not form a modular pair. Let $h_1 \neq h_2$ be two hyperplanes in the upset, $l = h_1 \wedge h_2$ and $p \neq q$ be two points on l . Then, since the lines partition E and are pairwise coplanar, necessarily $l_p \leq h_1$ and $l_q \leq h_2$ implying $l_p = l_q = l$. Finally, consider a hyperplane h and a line l . If they are a modular pair then they

must intersect in a point r , hence $l = l_r$. Furthermore, h must cover a line l_p , hence $h = l_p \vee r$ and since l_p and l_r are disjoint coplanar we must have $h = l_p \vee l_r$. Thus \mathcal{M} is a modular cut defining a point extension where l_1 and l_2 intersect.

(ii) Let p denote the new point and l a line containing p . Let q be another point on l . Then the restriction of l to M is the line l_q from Lemma 3.

(iii) By (i) there exist l_3 and l_4 in the partition which are not coplanar. It remains to show, that $\{l_3, l_4\} \subseteq \Delta$.

Assume w.l.o.g. that $l_3 = l_r \in \Sigma$ and hence $l_4 = l_q \in \Delta$. Since l_{p^*} and l_3 are coplanar we conclude $l_{p^*} \neq l_4$. If l_{p^*} and l_4 are not coplanar, we replace l_3 by l_{p^*} and are done, hence we may assume that they are coplanar. The hyperplanes $l_4 \vee r$, $l_{p^*} \vee r$ intersect in the line $l'_3 = (l_4 \vee r) \wedge (l_{p^*} \vee r)$. Assuming $l'_3 \leq e$ yields $l'_3 = (l_4 \vee r) \wedge (l_{p^*} \vee r) \wedge (l_1 \vee l_2) = l_r = l_3$ contradicting l_3 and l_4 being not coplanar. Hence l'_3 intersects e only in r . Furthermore, by Proposition 2 l'_3 must be disjoint from l_{p^*} and l_4 . Choose p' on l'_3 but not on e and define $l''_3 := l_{p'} \in \Delta$. We claim that $l_{p'}$ must be non coplanar with at least one of l_{p^*} or l_4 . Otherwise, we had

$$l''_3 = (l_{p^*} \vee l_{p'}) \wedge (l_4 \vee l_{p'}) = (l_{p^*} \vee p') \wedge (l_4 \vee p') = (l_{p^*} \vee l'_3) \wedge (l_4 \vee l'_3) = l'_3$$

which is impossible, since $l''_3 \in \Delta$ is disjoint from e . \square

The absence of a configuration in Theorem 10 (iii) is called bundle condition in the literature.

Definition 2. A matroid M of rank at least 4 satisfies the bundle condition, if for any 4 disjoint lines l_1, \dots, l_4 of M , no three of them coplanar, the following holds: If five of the six pairs (l_i, l_j) are coplanar, then all pairs are coplanar.

Since a non modular pair of hyperplanes together with the full matroid always forms a modular cut which is not principal, OTE-matroids must be hypermodular. Hence, Theorem 10 has the following Corollary.

Corollary 2. Let M be a OTE-matroid of rank 4. If the bundle-condition in M holds, then M is modular.

5. EMBEDDING THEOREMS

With these results, we can prove a first embedding theorem. Assertion (iii) is a result of Kahn [4].

Theorem 11. Let M be a hypermodular rank-4 matroid with a finite or countably infinite groundset. Then M is embeddable in an OTE-matroid M^* of rank 4, where the restriction of any line of M^* still is a line in M . Furthermore:

- (i) M^* is finite if and only if M is finite.
- (ii) The simplification of M^*/p is isomorphic to the simplification of M/p for every $p \in M$.
- (iii) If M fulfills the bundle-condition then M^* is modular.

Proof. Let P be a list of all disjoint coplanar pairs of lines of M . Clearly, P is finite or countably infinite. We inductively define a chain of extensions $M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ as follows: Let $M_0 = M$, suppose M_{i-1} has already been defined for an $i \in \mathbb{N}$. Let l_{i1} and l_{i2} denote the pair of disjoint lines in the list at index i . If l_{i1} and l_{i2} are not intersectable in the matroid M_{i-1} , set $M_i = M_{i-1}$. Otherwise, let M_i be the single-point extension of M_{i-1} corresponding to the modular cut created by l_{i1} and l_{i2} in M_{i-1} .

By Theorem 10 (ii) the restriction of a line in M_{i+1} is a line in M_i and hence the same holds in M . Thus all matroids M_i are hypermodular of rank 4. Now let M^* be the set system (E^*, \mathcal{I}^*) where $\mathcal{I}^* \subseteq \mathcal{P}(E^*)$, $E^* = \bigcup_{i=0}^{\infty} (E(M_i))$ and $I \in \mathcal{I}^*$ if and only if I is independent in some M_i . Clearly, \mathcal{I}^* satisfies the independence axioms of matroid theory. We call M^* the *union of the chain of extensions*. M^* is hypermodular of rank 4 and has no new lines as well.

Assume there were a modular cut \mathcal{M}^* in M^* which is not principal. By Proposition 5 it contains a pair of disjoint coplanar lines. The restriction of this pair in M has been on the list, say with index i . Their modular cut \mathcal{M}_i generated M_i must contain $cl_{M_i}(\emptyset)$, otherwise the lines would intersect in M_{i+1} , hence also in M^* . Since $\{cl_{M^*}(X) | X \in \mathcal{M}_i\} \subseteq \mathcal{M}^*$ we also must have $cl_{M^*}(\emptyset) \in \mathcal{M}^*$, a contradiction to \mathcal{M}^* not being principal. Thus, M^* is OTE. If M is finite, so is the list P and hence M^* proving (i).

It suffices to show that every point $q \in (M^*/p) \setminus (M/p)$ is parallel to a point in M/p . As the restriction of line spanned by p and q in M^* is a line in M it contains a point different from p and (ii) follows. Finally, (iii) is Corollary 2. \square

From this embedding theorem follows as a corollary:

Corollary 3. *Kantor's conjecture is reducable to the rank-4 case.*

Proof. Assume Kantor's conjecture holds for rank-4 matroids. Let M be a finite hypermodular matroid of rank $n > 4$. All contractions of M by a flat of rank $n - 4$ are finite hypermodular matroids of rank 4, hence are embeddable into a modular matroid. Using Theorem 11 it is easy to see, that these contractions are also *strongly embeddable* (as defined in [5], Definition 2) into a modular matroid. Hence the matroid M satisfies the assumptions of Theorem 2 in [5], and thus is embeddable into a modular matroid, implying the general case of Kantor's Conjecture. \square

We have a second embedding theorem:

Theorem 12. *Let M be a matroid of finite rank on a set E where E is finite or countably infinite. Then M is embeddable in an OTE-matroid of the same rank.*

Proof. We proceed similar to the proof of Theorem 11. Let P be the list of all intersectable non-modular pairs of M . We build an extension-chain $M = M_0 \subseteq M_1 \dots$, where each matroid M_{i+1} is the extension of M_i , where the modular defect of the i -th pair on the list can no longer be decreased. Let M^* be the union of the extension chain like in the proof before. Then M^* is a matroid of finite rank with a finite or countably infinite ground set. If there still are non-modular pairs in M^* we repeat the process and obtain M_1^* . This yields an extension chain $M^* \subseteq M_1^* \subseteq M_2^* \subseteq \dots$. Let M^{**} be the union of that extension chain. Clearly, M^{**} is a matroid. We claim it is OTE. For assume it had a non-trivial modular cut generated by a pair of intersectable flats f_1, f_2 . Since their rank is finite, there exists an index k such that the matroid M_k^* contains a basis of f_1 as well as of f_2 . But then in the matroid M_{k+1}^* the pair would not be intersectable anymore and we get a contradiction. Thus, M^{**} is an OTE-matroid. \square

We have a similar result for hypermodular matroids:

Theorem 13. *Every matroid M of finite rank r with maximal countably infinite groundset is embeddable in a infinite hypermodular matroid M^* of rank r .*

Proof. The proof mimics the one of Theorem 12, except that we have only the non-modular pairs of hyperplanes in the list. This generalizes the technique of *free closure* of rank-3 matroids and it is not difficult to show (see e.g. Kantor [5], Example 5) that in case M is non-modular (hence $r \geq 3$), the contractions of M^{**} by a flat of rank $r-3$ in M^{**} are infinite projective non-Desarguesian planes and hence M^{**} must be infinite, too. \square

6. ON THE NON-EXISTENCE OF CERTAIN MODULAR PAIRS IN EXTENSIONS OF OTE-MATROIDS

In order to prove that the proper amalgam exists for any two extensions of a finite rank-4 OTE-matroid we need some technical lemmas. We will show that certain modular pairs cannot exist in extensions of rank-4 OTE-matroids. We need some preparations for that.

Proposition 6. *Let M be matroid with groundset T , let (X, Y) be a modular pair of subsets of T and let $Z \subseteq X \setminus Y$. Then $(X \setminus Z, Y)$ is a modular pair, too.*

Proof. Submodularity implies $r(X \cup Y) - r(X) \leq r((X \setminus Z) \cup Y) - r(X \setminus Z)$. Using modularity of (X, Y) we find

$$\begin{aligned} r(X \setminus Z) + r(Y) &= r(X \cup Y) + r((X \setminus Z) \cap Y) - r(X) + r(X \setminus Z) \\ &\leq r((X \setminus Z) \cup Y) + r((X \setminus Z) \cap Y) \end{aligned}$$

and another application of submodularity implies the assertion. \square

By (D) we abbreviate the following list of assumptions:

- M is a matroid with groundset T and rank function r .
- M' is an extension of M with rank function r' and groundset E' .
- $X, Y \subseteq E'$ are subsets of E' such that $X \cap T = l_X, Y \cap T = l_Y$ are two disjoint coplanar lines in M and
- $X \cap Y$ is a flat in M' .

Proposition 7. *Assume (D) and, furthermore, that $(X \setminus T) \subseteq Y$ and that (X, Y) is a modular pair of sets in M' . Then*

$$x \notin cl_{M'}(Y) \quad \text{for all } x \in l_X.$$

Proof. Assume to the contrary that there exists $x \in l_X$ with $x \in cl_{M'}(Y)$. Then coplanarity of l_X and l_Y implies

$$X \cap T = l_X \subseteq l_X \vee l_Y = x \vee l_Y \subseteq cl_{M'}(Y).$$

Hence $X \subseteq cl_{M'}(Y)$, implying $r'(Y) = r'(X \cup Y)$ and modularity of (X, Y) yields $r'(X) = r'(X \cap Y)$, a contradiction, because $X \cap Y$ is a flat in M' and a proper subset of X . \square

Lemma 4. *Assume (D) and that M is of rank 4 (the rank of M' may be larger) and, furthermore,*

- (X, Y) is supposed to be a modular pair of sets in M' with $(X \setminus T) \subseteq Y$ and $T \not\subseteq cl_{M'}(X \cup Y)$ and
- $l' \subseteq T$ is a line disjoint coplanar to l_X and l_Y , not lying in $l_X \vee l_Y$.

Then $X' = (X \setminus T) \cup l'$ implies $r'(X') = r'(X)$.

Proof. Choose $x \in l_X$ and $x' \in l' = X' \cap T$. Because l_X and l_Y are coplanar and $(X \setminus T) \subseteq Y$ we conclude $cl_{M'}(\{x\} \cup Y) = cl_{M'}(X \cup Y)$. Similarly, we get $cl_{M'}(\{x'\} \cup Y) = cl_{M'}(X' \cup Y)$.

By assumption M , being of rank 4, is spanned by l', l_X and l_Y and hence $T \subseteq cl_{M'}(\{x'\} \cup \{x\} \cup Y)$. The assumption $x' \in cl_{M'}(\{x\} \cup Y)$ implies that $T \subseteq cl_{M'}(\{x\} \cup Y) = cl_{M'}(X \cup Y)$, contradicting the assumptions, thus $x' \notin cl_{M'}(\{x\} \cup Y)$. In particular $x' \notin cl_{M'}(X)$.

Assuming $x \in cl_{M'}(\{x'\} \cup Y)$ Proposition 7 yields $x \notin cl_{M'}(Y)$. Now, using the exchange-axiom of the closure-operator, we find $x' \in cl_{M'}(\{x\} \cup Y)$, contradicting the above. Hence also $x \notin cl_{M'}(\{x'\} \cup Y) = cl_{M'}(X' \cup Y)$. In particular $x \notin cl_{M'}(X')$.

The choice of x and x' implies $cl_M(l_X \cup \{x'\}) = cl_M(l' \cup \{x\})$ and using $X \setminus T = X' \setminus T$ we obtain $cl_{M'}(X \cup \{x'\}) = cl_{M'}(X' \cup \{x\})$. We conclude

$$r'(X') + 1 = r'(X' \cup \{x\}) = r'(X \cup \{x'\}) = r'(X) + 1. \quad \square$$

Lemma 5. *Assume (D), M is a rank-4 OTE-matroid and $(X \setminus T) \subseteq Y$, $(Y \setminus T) \subseteq X$ and $T \not\subseteq cl_{M'}(X \cup Y)$. Then $\delta_{M'}(X, Y) > 0$.*

Proof. M is hypermodular, OTE and of rank 4. Hence by Theorem 10 (iii) it has two lines l_1 and l_2 that span M but are both disjoint coplanar to l_X and l_Y .

Assume that $\delta_{M'}(X, Y) = 0$. Let $X' = (X \setminus T) \cup l_1$ and $Y' = (Y \setminus T) \cup l_2$. Then by Lemma 4

$$(1) \quad r'(X') = r'(X) \text{ and } r'(Y') = r'(Y).$$

Since $T \subseteq cl_M(l_1, l_2) \subseteq cl_{M'}(X' \cup Y')$ and $T \not\subseteq cl_{M'}(X \cup Y)$ we get

$$(2) \quad r'(X' \cup Y') = r'(X' \cup Y' \cup T) = r'(X \cup Y \cup T) > r'(X \cup Y).$$

By definition $X' \cap Y' = (X \setminus T) \cap (Y \setminus T) = X \cap Y$ and hence by sumodularity

$$\begin{aligned} r'(X \cup Y) + r'(X \cap Y) &\stackrel{(2)}{<} r'(X' \cup Y') + r'(X' \cap Y') \\ &\leq r'(X') + r'(Y') \stackrel{(1)}{=} r'(X) + r'(Y) \end{aligned}$$

hence $\delta_{M'}(X, Y) > 0$, a contradiction. \square

We come to the main result of this section.

Theorem 14. *Let M be a rank-4 OTE-matroid with groundset T and M' an extension of M with ground set E' . Let $X, Y \subseteq E'$ be sets such that $X \cap Y$ is a flat in M' and the restrictions $l_X = X \cap T$ and $l_Y = Y \cap T$ are disjoint coplanar lines in M . Then $T \not\subseteq cl_{M'}(X \cup Y)$ implies that (X, Y) is not a modular pair in M' .*

Proof. Assume to the contrary that (X, Y) were a modular pair in M' . Let $X' = (X \cap T) \cup (X \cap Y)$ and $Y' = (Y \cap T) \cup (X \cap Y)$. Applying Proposition 6 twice, we find that the pair (X', Y') is modular in M' , too, and satisfies the assumptions of Lemma 5 yielding the required contradiction. \square

By contraposition we get

Corollary 4. *Let M be a rank-4 OTE-matroid with groundset T and M' an extension of M . Let (X, Y) be a modular pair of flats in M' such that $(X \cap T, Y \cap T)$ is a non-modular pair in M . Then $T \subseteq cl_{M'}(X \cup Y)$.*

Regarding the case that $(X \cap T, Y \cap T)$ is a disjoint line-plane pair, we show the following.

Lemma 6. *Let M be a rank-4 OTE-matroid with groundset T and rank function r and let M' be an extension of M with groundset E' and rank function r' . Assume that $X, Y \subseteq E'$ are sets such that $X \cap T = e_X$ is a plane, $Y \cap T = l_Y$ a line disjoint from e_X in M , and that $X \cap Y$ is a flat in M' . Assume that there exists a line $l_X \subseteq e_X$ coplanar with l_Y such that $r'((X \cap Y) \cup e_X) = r'((X \cap Y) \cup l_X) + 1$. Then $\delta_{M'}(X, Y) > 0$.*

Proof. Assume, for a contradiction, (X, Y) were a modular pair of flats in M' and let $X' = (X \cap Y) \cup e_X$. Since $X' = X \setminus Z$ with $Z = (X \setminus Y) \setminus e_X \subseteq X \setminus Y$ we find that by Proposition 6 (X', Y) is a modular pair in M' , too. Let $X'' = (X' \setminus T) \cup l_X$. By assumption $r'(X') = r'(X'') + 1$ and $X'' \cap T$ is a line disjoint from and coplanar to l_Y . Moreover $X'' \cap Y = X' \cap Y$, thus $X'' \cap Y$ is a flat in M' . Furthermore submodularity implies $r'(X' \cup Y) \leq r'(X'' \cup Y) + 1$. Because (X', Y) is a modular pair we obtain:

$$\begin{aligned} r'(X'' \cup Y) + 1 + r'(X'' \cap Y) &\geq r'(X' \cup Y) + r'(X' \cap Y) \\ &= r'(X') + r'(Y) = r'(X'') + 1 + r'(Y) \end{aligned}$$

and, again submodularity of r' implies that equality must hold throughout. Hence (X'', Y) is a modular pair and

$$r'(X'' \cup Y) + 1 = r'(X' \cup Y) = r'(X' \cup Y \cup T) = r'(X'' \cup Y \cup T)$$

implying $T \not\subseteq cl_{M'}(X'' \cup Y)$. The pair (X'', Y) now contradicts Theorem 14. \square

7. THE PROPER AMALGAM

We prove Theorem 5 by constructing the so called *proper amalgam* of two extensions of a rank-4 OTE-matroid. In this section we define this amalgam and we analyse some of its properties. Throughout, if not mentioned otherwise, we assume the following situation.

M is a matroid with groundset T and rank function r and M_1 and M_2 are extensions of M with the groundsets E_1 resp. E_2 and rank functions r_1 resp. r_2 , where $E_1 \cap E_2 = T$ and $E_1 \cup E_2 = E$. All matroids are of finite rank with finite or countably infinite ground set. We define two functions $\eta : \mathcal{P}(E) \mapsto \mathbb{Z}$ und $\xi : \mathcal{P}(E) \mapsto \mathbb{Z}$ by

$$\eta(X) = r_1(X \cap E_1) + r_2(X \cap E_2) - r(X \cap T)$$

$$\text{and } \xi(X) = \min\{\eta(Y) : Y \supseteq X\}.$$

The following is immediate:

Proposition 8. *The function ξ is subcardinal, finite and monotone, it holds:*

R1 : $0 \leq \xi(X) \leq |X|$, for all $X \subseteq E$.

R1a : For all $X \subseteq E$ there exist an $X' \subseteq X, |X'| < \infty$, such that $\xi(X) = \xi(X')$.

R2 : For all $X_1 \subseteq X_2 \subseteq E$ we have $\xi(X_1) \leq \xi(X_2)$.

Moreover $\xi(X) \leq \eta(X)$ for all $X \subseteq E$.

If ξ is submodular on $\mathcal{P}(E)$, then ξ is the rank function of an amalgam of M_1 and M_2 along M (see eg. [7], Proposition 11.4.2). This amalgam, if it exists, is called the *proper amalgam of M_1 and M_2 along M* .

Now let $\mathcal{L}(M_1, M_2)$ be the set of all subsets X of E , so that $X \cap E_1$ and $X \cap E_2$ are flats in M_1 resp. M_2 . Then it is easy to see that $\mathcal{L}(M_1, M_2)$ with the inclusion-ordering is a complete lattice of subsets of E . Let $\wedge_{\mathcal{L}}$ and $\vee_{\mathcal{L}}$ be the meet resp. the join of this lattice. Clearly, for two sets $X, Y \in \mathcal{L}(M_1, M_2)$ we have $X \wedge_{\mathcal{L}} Y = X \cap Y$ and $X \vee_{\mathcal{L}} Y \supseteq X \cup Y$. We need two results from [7].

Lemma 7 (see [7] Prop. 11.4.5.). *For all $X \subseteq E$*

$$\xi(X) = \min\{\eta(Y) : Y \in \mathcal{L}(M_1, M_2) \text{ and } Y \supseteq X\}.$$

Lemma 8 (see [7] Lemma 11.4.6.). *Let $Y \subseteq E$ and Z be the smallest element of $\mathcal{L}(M_1, M_2)$ containing Y , then $\eta(Z) \leq \eta(Y)$ holds.*

The literature seems to give a proof of these results only for the case of a finite ground set. We present a proof for possibly infinite ground sets for completeness.

Proof. For all $X \subseteq E$ let $\phi_1(X) = \text{cl}_1(X \cap E_1) \cup (X \cap E_2)$ and $\phi_2(X) = (X \cap E_1) \cup \text{cl}_2(X \cap E_2)$. Then

$$\phi_1(X) \cap E_2 = (\text{cl}_1(X \cap E_1) \cap T) \cup (X \cap E_2) = (\phi_1(X) \cap T) \cup (X \cap E_2)$$

as well as

$$X \cap T = (\phi_1(X) \cap T) \cap (X \cap E_2).$$

Hence submodularity of r_2 implies

$$r_2(\phi_1(X) \cap T) + r_2(X \cap E_2) \geq r_2(\phi_1(X) \cap E_2) + r_2(X \cap T)$$

and thus

$$r_2(\phi_1(X) \cap E_2) - r_2(\phi_1(X) \cap T) \leq r_2(X \cap E_2) - r_2(X \cap T).$$

Since $r_1(\phi_1(X) \cap E_1) = r_1(X \cap E_1)$ we compute

$$\begin{aligned} \eta(\phi_1(X)) &= r_1(\phi_1(X) \cap E_1) + r_2(\phi_1(X) \cap E_2) - r(\phi_1(X) \cap T) \\ &\leq r_1(X \cap E_1) + r_2(X \cap E_2) - r(X \cap T) = \eta(X). \end{aligned}$$

Using symmetry we derive

$$\eta(\phi_i(X)) \leq \eta(X) \text{ for all } X \subseteq E \text{ and } i = 1, 2.$$

Now let Z be the minimal element in $\mathcal{L}(M_1, M_2)$ such that $Y \subseteq Z$ and choose $Y \subseteq W \subseteq Z$ maximal with

$$\eta(W) \leq \eta(Y).$$

From $Y \subseteq W \subseteq \phi_i(W) \subseteq Z$ and $\eta(\phi_i(W)) \leq \eta(W)$ follows $\phi_i(W) = W$ for $i = 1, 2$ and hence $W = Z \in \mathcal{L}(M_1, M_2)$ and Lemma 8 follows, also implying Lemma 7. \square

Note that the proof of this lemma and the case (R1a) of Proposition 8 imply that Theorem 1 holds for infinite matroids of finite rank as well. Now we generalize a result of Ingleton (cf. [7], Theorem 11.4.7):

Theorem 15. *Assume that for all pairs (X, Y) of sets of $\mathcal{L}(M_1, M_2)$ either η or ξ or both are submodular. Then ξ is submodular on $\mathcal{P}(E)$ and the proper amalgam of M_1 and M_2 along M exists.*

Proof. Let $X_1, X_2 \subseteq E$. By Lemma 7 we find $Y_i \in \mathcal{L}(M_1, M_2)$, such that $X_i \subseteq Y_i$ and $\xi(X_i) = \eta(Y_i)$ for $i = 1, 2$. From $\eta(Y_i) = \xi(X_i) \leq \xi(Y_i) \leq \eta(Y_i)$ we conclude that $\xi(X_i) = \xi(Y_i) = \eta(Y_i)$. Hence, by assumption either η or ξ or both are submodular on the pair of flats (Y_1, Y_2) . Furthermore, $X_1 \cap X_2 \subseteq Y_1 \cap Y_2 = Y_1 \wedge_{\mathcal{L}} Y_2$ and $X_1 \cup X_2 \subseteq Y_1 \cup Y_2 \subseteq Y_1 \vee_{\mathcal{L}} Y_2$. Hence, by Proposition 8

$$\xi(X_1 \cap X_2) + \xi(X_1 \cup X_2) \leq \xi(Y_1 \wedge_{\mathcal{L}} Y_2) + \xi(Y_1 \vee_{\mathcal{L}} Y_2).$$

Thus, if η is submodular on (Y_1, Y_2)

$$\begin{aligned} \xi(X_1 \cap X_2) + \xi(X_1 \cup X_2) &\leq \eta(Y_1 \wedge_{\mathcal{L}} Y_2) + \eta(Y_1 \vee_{\mathcal{L}} Y_2) \\ &\leq \eta(Y_1) + \eta(Y_2) = \xi(X_1) + \xi(X_2) \end{aligned}$$

and otherwise

$$\xi(X_1 \cap X_2) + \xi(X_1 \cup X_2) \leq \xi(Y_1) + \xi(Y_2) = \xi(X_1) + \xi(X_2).$$

Hence ξ is submodular on $\mathcal{P}(E)$ and the proper amalgam exists. \square

Lemma 8 immediately yields

Lemma 9. *Let X, Y be of $\mathcal{L}(M_1, M_2)$, then it holds $\eta(X \cup Y) \geq \eta(X \vee_{\mathcal{L}} Y)$. Moreover we have: $\xi(X \cup Y) = \xi(X \vee_{\mathcal{L}} Y)$.*

We finish this section with a small lemma.

Lemma 10. *Additionally to the general assumptions let M be of rank 4. Let $X \in \mathcal{L}(M_1, M_2)$ with $r(X \cap T) \geq 2$. Then $\xi(X) = \eta(X)$.*

Proof. Assume there exists $Y \supseteq X$ such that $\xi(X) = \eta(Y) < \eta(X)$. Then $r(Y \cap T) > r(X \cap T)$. Hence we find $t \in (Y \cap T) \setminus X$, and because $X \cap E_1, X \cap E_2$ and $X \cap T$ are flats we get

$$\begin{aligned} \eta(X \cup \{t\}) &= r_1((X \cup \{t\}) \cap E_1) + r_2((X \cup \{t\}) \cap E_2) - r((X \cup \{t\}) \cap T) \\ &= r_1(X \cap E_1) + 1 + r_2(X \cap E_2) + 1 - r(X \cap T) - 1 = \eta(X) + 1. \end{aligned}$$

But since M is of rank 4 and $r((X \cup \{t\}) \cap T) \geq 3$, the decrease of η for supersets of $X \cup \{t\}$ is bounded by 1 and thus $\eta(Y) \geq \eta(X \cup \{t\}) - 1 = \eta(X)$, a contradiction. \square

8. PROOF OF THEOREM 5

Our proof of Theorem 5 may be considered a generalization of the proof of Proposition 11.4.9. in [7]. Oxley refers to unpublished results of A.W. Ingleton. We start with a lemma.

Lemma 11. *Let M be a rank-4-OTE-matroid with ground set T . Let M_1 and M_2 be two extensions of M with the ground sets E_1, E_2 and rank functions r_1, r_2 . Let $E_1 \cap E_2 = T$ and $E_1 \cup E_2 = E$ and let η, ξ and $\mathcal{L}(M_1, M_2)$ be defined as in Section 7.*

Let (X, Y) be a pair of elements of $\mathcal{L}(M_1, M_2)$ that violates the submodularity of η . Then

- (i) $\eta(X) + \eta(Y) - \eta(X \cap Y) - \eta(X \cup Y) = \delta(X \cap E_1, Y \cap E_1) + \delta(X \cap E_2, Y \cap E_2) - \delta(X \cap T, Y \cap T) = -1$.
- (ii) $(X \cap E_i, Y \cap E_i)$ is a modular pair in M_i for $i = 1, 2$.
- (iii) $(X \cap T, Y \cap T)$ are two disjoint coplanar lines or a disjoint line-plane pair in M .
- (iv) $\eta(X) = \xi(X)$ and $\eta(Y) = \xi(Y)$.

Proof. (i) A straightforward computation yields the first equality. The second one follows from the fact that the modular defect in a rank-4-matroid is bounded by 1.
(ii) and (iii) are immediate from (i) and (iv) follows from Lemma 10. \square

Lemma 12. *Under the assumptions of Lemma 11 let (X, Y) be a pair of elements of $\mathcal{L}(M_1, M_2)$ such that the submodularity of η in $\mathcal{L}(M_1, M_2)$ is violated, and $\xi(X \cup Y) < \eta(X \cup Y)$ or $\xi(X \cap Y) < \eta(X \cap Y)$. Then ξ is submodular for (X, Y) in $\mathcal{L}(M_1, M_2)$.*

Proof. Recall that $\xi(X \cup Y) \leq \eta(X \cup Y)$ and $\xi(X \cap Y) \leq \eta(X \cap Y)$ and $\xi(X \cap Y) = \xi(X \wedge_{\mathcal{L}} Y)$ as well as $\xi(X \cup Y) = \xi(X \vee_{\mathcal{L}} Y)$ by Lemma 9. Moreover by Lemma 11 (iv) $\eta(X) = \xi(X)$ and $\eta(Y) = \xi(Y)$. Altogether this implies

$$\begin{aligned} & \xi(X) + \xi(Y) - \xi(X \wedge_{\mathcal{L}} Y) - \xi(X \vee_{\mathcal{L}} Y) \\ &= \xi(X) + \xi(Y) - \xi(X \cap Y) - \xi(X \cup Y) \\ &> \eta(X) + \eta(Y) - \eta(X \cap Y) - \eta(X \cup Y) = -1 \end{aligned}$$

proving the assertion. \square

We are now ready to tackle the proof of Theorem 5 which is an immediate consequence of the following:

Theorem 16. *Let M be a rank-4 OTE-matroid. Then for any pair of extensions of M the proper amalgam exists.*

Proof. Let T denote the ground set of M and M_1, M_2 be two extensions of M with ground sets E_1, E_2 and rank functions r_1, r_2 , such that $E_1 \cap E_2 = T$ and $E_1 \cup E_2 = E$. We show that for these two extensions the proper amalgam exists. Let η and ξ be defined as in the previous section. By Lemma 15 it suffices to show that for each pair (X, Y) of elements of $\mathcal{L}(M_1, M_2)$ either η or ξ is submodular.

We do a case checking for all possible pairs (X, Y) of sets of $\mathcal{L}(M_1, M_2)$ where the submodularity of η could be violated, and show that $\xi(X \cup Y) < \eta(X \cup Y)$ or $\xi(X \cap Y) < \eta(X \cap Y)$ and hence (by Lemma 12) ξ is submodular on (X, Y) .

By Lemma 11 $(X \cap E_i, Y \cap E_i)$ are modular pairs of flats in M_i for $i = 1, 2$ and $(X \cap T, Y \cap T)$ is a pair of disjoint coplanar lines or a disjoint line-plane-pair.

disjoint coplanar lines: Assume $X \cap T = l_X$ and $Y \cap T = l_Y$ are two disjoint coplanar lines. By Theorem 14 the fact that $(X \cap E_i, Y \cap E_i)$ are modular pairs for $i = 1, 2$ implies that $T \subseteq cl_{M_i}((X \cup Y) \cap E_i)$ for $i = 1, 2$. Let $t \in T \setminus cl_M(l_X \cup l_Y)$. Then

$$\begin{aligned} & \eta(X \cup (Y \cup \{t\})) \\ &= r_1((X \cup (Y \cup \{t\})) \cap E_1) + r_2((X \cup (Y \cup \{t\})) \cap E_2) \\ &\quad - r((X \cup (Y \cup \{t\})) \cap T) \\ &= r_1((X \cup Y) \cap E_1) + r_2((X \cup Y) \cap E_2) - r((X \cup Y) \cap T) - 1 \\ &= \eta(X \cup Y) - 1. \end{aligned}$$

Hence $\xi(X \cup Y) < \eta(X \cup Y)$.

disjoint point-line pair: Assume $X \cap T = e_X$ is a plane and $Y \cap T = l_Y$ a line disjoint from e_X . By Lemma 6 for every line $l \subseteq e_X$ such that $r(l \vee l_Y) = 3$ we must have

$$(3) \quad r_i((X \cap Y \cap E_i) \cup e_X) = r_i((X \cap Y \cap E_i) \cup l) \text{ for } i = 1, 2.$$

Choose a point $p_1 \in e_X$. Since M must be hypermodular $l_X = (e_X \wedge (l_Y \vee p_1))$ is a line and $p_1 \in l_X$. Since $Y \cap E_1$ is a flat not containing p_1 and $X \cap Y \cap E_1$ is a flat disjoint from T we have

$$(4) \quad r_1((Y \cup \{p_1\}) \cap E_1) = r_1(Y \cap E_1) + 1$$

$$(5) \quad r_1((X \cap Y \cap E_1) \cup \{p_1\}) = r_1(X \cap Y \cap E_1) + 1.$$

Choose a second point $p_2 \in l_X$ such that $p_2 \neq p_1$. Since l_X and l_Y are coplanar, we obtain

$$p_2 \in l_X \subseteq p_1 \vee l_Y = p_1 \vee (Y \cap T) \subseteq \text{cl}_{M_1}(\{p_1\} \cup (Y \cap E_1))$$

and thus

$$(6) \quad r_1((Y \cup l_X) \cap E_1) = r_1((Y \cup \{p_1, p_2\}) \cap E_1) = r_1((Y \cup \{p_1\}) \cap E_1).$$

Furthermore, since $\{p_1, p_2\} \subseteq l_X \subseteq X$:

$$(7) \quad r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) = r_1((X \cup Y) \cap E_1)$$

Using these equations and the modularity of $(X \cap E_1, Y \cap E_1)$ in M_1 we compute

$$\begin{aligned} & r_1(X \cap E_1) + r_1((Y \cup \{p_1, p_2\}) \cap E_1) \\ \stackrel{(6)}{=} & r_1(X \cap E_1) + r_1((Y \cup \{p_1\}) \cap E_1) \\ \stackrel{(4)}{=} & r_1(X \cap E_1) + r_1(Y \cap E_1) + 1 \\ \stackrel{(Mod.)}{=} & r_1((X \cup Y) \cap E_1) + r_1(X \cap Y \cap E_1) + 1 \\ \stackrel{(7)}{=} & r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) + r_1(X \cap Y \cap E_1) + 1 \\ \stackrel{(5)}{=} & r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) + r_1((X \cap Y \cap E_1) \cup \{p_1\}) \\ \leq & r_1((X \cup Y \cup \{p_1, p_2\}) \cap E_1) + r_1((X \cap Y \cap E_1) \cup \{p_1, p_2\}) \end{aligned}$$

By submodularity of r_1 the last inequality must hold with equality and hence

$$(8) \quad r_1((X \cap Y \cap E_1) \cup l_X) = r_1((X \cap Y \cap E_1) \cup \{p_1\}).$$

By symmetry (5) and (8) are also valid for r_2 and E_2 . Recalling that $X \cap Y \cap T = \emptyset$ we compute

$$\begin{aligned}
\eta((X \cap Y) \cup e_X) &= \sum_{i=1}^2 r_i((X \cap Y \cap E_i) \cup e_X) - r(e_X) \\
&\stackrel{(3)}{=} \sum_{i=1}^2 r_i((X \cap Y \cap E_i) \cup l_X) - 3 \\
&\stackrel{(8)}{=} \sum_{i=1}^2 r_i((X \cap Y \cap E_i) \cup \{p_1\}) - 3 \\
&\stackrel{(5)}{=} \sum_{i=1}^2 (r_i(X \cap Y \cap E_i) + 1) - r(X \cap Y \cap T) - 3 \\
&= \eta(X \cap Y) - 1.
\end{aligned}$$

Hence $\xi(X \cap Y) < \eta(X \cap Y)$.

□

9. CONCLUSION

Now if we put the embedding theorems together with Theorem 5 we get the equivalence of three conjectures:

Theorem 17. *The following statements are equivalent:*

- (i) *Every finite matroid, which is sticky, is modular. (SMC)*
- (ii) *Every finite hypermodular matroid is embeddable in a modular matroid. (Kantor's Conjecture)*
- (iii) *Every finite OTE-matroid is modular.*

Proof. (i) \Rightarrow (ii) These two statements can be reduced to the rank-4 case, see Theorem 2 and Corollary 3. Now consider a finite hypermodular rank-4 matroid M . Because of Theorem 11 it can be embedded into a finite rank-4 OTE-matroid M' , which is sticky due to Theorem 5. If (i) holds then M' is modular and M can be embedded into a modular matroid and (ii) holds.

(ii) \Rightarrow (iii) Let M be a finite OTE-matroid. It is also hypermodular. If (ii) holds, it is embeddable into a modular matroid. Since M is OTE, it must itself already be modular.

(iii) \Rightarrow (i) Let M be a finite sticky matroid. Because of Theorem 3 it must be an OTE-matroid and, if (iii) holds, must be modular and (i) holds. □

A slightly weaker conjecture than the (SMC) in the finite case, which could also hold in the infinite case, is the generalization of Theorem 5 to arbitrary rank.

Conjecture 4. *A matroid is sticky if and only if it is an OTE-matroid.*

Our proof of Theorem 5 frequently uses the fact, that we are dealing with rank 4 matroids. We think there is a way to avoid Lemma 10, but the case checking in the proof of Theorem 16 seems to become tedious even for ranks only slightly larger than 4. Moreover, we need a generalization of Theorem 10 (iii) in order to generalize Lemma 4.

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